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Global action-angle variables and the characterization of one degree of freedom rational Hamiltonians

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Abstract. We analyse the problem of constructing global complex action-angle variables for a class of Hamiltonians $\mathcal{H}(p, q)$ rational in q and quadratic in p.

We give a complete description of the monodromy of the action variables in terms of subgroups of unimodular transformations associated to the singular energy points.

We also consider a peculiar subclass of Hamiltonian systems which are reducible to algebraically complete integrable systems and show that the reduction preserves the symplectic structure. As a consequence the monodromy of the action variable is induced by the one of the reduced system.

Finally, we consider the problem of defining global angle variables. We propose to define a set of angle variables which correspond to a group of Poisson structures associated to $\mathcal{H}$.

1. Introduction

In this paper we construct complex action-angle variables for Hamiltonian systems with one degree of freedom to which we can apply the powerful techniques of algebraic geometry, but that are not, in general, algebraically completely integrable (see [4, 22]). For the construction of action variables of ACI systems see [12, 16, 23].

Real action-angle variables are defined locally on compact invariant surfaces of Liouville-integrable Hamiltonian systems (for definitions see [5]). The definition proposed here of local complex action-angle variables is a straightforward generalization of the real one and, in particular, it generalizes the definition of action-angle variables to the case of real unbounded motions since we consider compactified surfaces associated to $\mathcal{H} = \mathcal{E}$.

Our main interest, however, concerns global properties of the complex action-angle variables, since our motivation for introducing them and studying their properties for one degree of freedom systems is to establish integrability criteria for higher degrees of freedom systems. We remark that the study of monodromy properties of the action is preliminar to integrability criteria for higher dimensional systems. Indeed, the strong connection between real and complex dynamics has been widely pointed out in literature [7–10, 12, 20, 23–25] and, in particular, Ziglin [24, 25] showed that the monodromy of the action variable of an unperturbed system sets conditions on the integrability of a generic perturbation of the Hamiltonian. We recall that there is also a global problem for real action-angle variables [11, 15], which is not considered here.

The local action depends on the close cycle along which it is computed and, when the system is not ACI, the local variable is not well defined globally. To solve these problems, we introduce a vector of angle variables and a complex matrix of actions computed on a basis of cycles. In this way, we get a natural setting for characterizing their global properties.

The main result is the complete classification of the monodromy of the complex action $\mathcal{I}$ with respect to energy $\mathcal{E}$. In particular we show that the analytic properties of the action matrix depend on subgroups of unimodular transformations of the associated period matrix and, consequently, the action may be continued analytically but not single valuedly in $\mathcal{E}$ allowing for a complete classification of the singular energy points which correspond to singular curves (separatrices).

We consider one-degree-of-freedom systems, not only because their complex structure is quite interesting, but in view of applications to higher-degree-of-freedom systems. Indeed the approach considered here may be generalized to the corresponding multidimensional situation thanks to the powerful properties of Abelian varieties and Abelian integrals. Moreover, it is possible to apply the results presented here to Hamiltonians obtained by perturbations of (1). Indeed, in some classes of perturbed non-integrable Hamiltonians the local singularity structure is asymptotically conjugated to that of the corresponding integrable unperturbed system (for results in this direction see [2] and references therein).

Finally, we remark that there are physically interesting systems—such as the Lagrange top whose flow (see [20]) linearizes on elliptic curves—which may be studied along the way proposed here. Indeed, we may study the analytic dependence of the action variable associated to the elliptic curve in function of the three integrals of the system (the energy, the angular momentum with respect to the figure axis and the direction of gravity) which appear as parameters of the elliptic curve after the double reduction procedure. Of course this description of the system will be quite complicated because each of these parameters is allowed to vary in $\mathbb{C}$.

In section 2 we introduce the local complex action angle variables, set conditions in order that the angle variable be a holomorphic $(1, 0)$-form and classify action variables according to their singularities on $\mathcal{R}_{\mathcal{E}}$. In section 3 we define the global action-angle variables and set, in particular, the monodromy properties of the action matrix in theorem 4. In section 4 we consider some examples. In section 5 we consider action-angle variables associated to reducible Hamiltonians, that is the exceptional but interesting case for $g > 1$ in which the local angle variable is well defined globally. We show that the rational transformation associated to the reduction induces a symplectic change of coordinates and, as a consequence, the monodromy properties of the reducible and reduced actions are the same. Finally, in section 6 we consider some examples of reducible Hamiltonians.

2. Local action-angle variables

We consider rational Hamiltonians

$$\mathcal{H}(q, p) = \tfrac{1}{2} V_0(q) p^2 + V_1(q) \tag{1}$$

where $(q, p) \in \mathbb{C}^2$, V_0, V_1 are rational in q. We suppose that $\mathcal{H}$ has a canonical Poisson structure. We shall release this condition in the next section where we introduce a family of Poisson structures associated to the global angle variables.

Equation (1) is integrable in Liouville sense, but in general it is not algebraically completely integrable (see [4, 22]). To the real bounded motions of (1), we may associate real action-angle variables

$$\mathcal{I} = \frac{1}{2\pi} \oint_{\gamma} p \, \mathrm{d}q \qquad \phi = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\mathcal{I}} \int \frac{\partial p}{\partial \mathcal{E}} \, \mathrm{d}q. \tag{2}$$

We are interested here in the properties of the action-angle variables in the complex domain (q, p) in the case in which the angle variable is a holomorphic $(1, 0)$-form. Let us denote

$$V_0(q) = \frac{V_0^N(q)}{V_0^D(q)} \qquad V_1(q) = \frac{V_1^N(q)}{V_1^D(q)} \tag{3}$$

where V_0^D, V_0^N (respectively V_1^D, V_1^N) are prime. Then,

$$\frac{\partial p}{\partial \mathcal{E}} \, \mathrm{d}q = \frac{Q(q)}{u} \, \mathrm{d}q \tag{4}$$

where

$$\begin{aligned} Q(q) &= \sqrt{V_0^D(q) W_1^D(q)} \\ u^2 &= 2(\mathcal{E} V_1^D(q) - V_1^N(q)) W_0^N(q) \end{aligned} \tag{5}$$

and $P(q)$ is the maximum common divisor between $V_0^N = P(q) W_0^N$ and $V_1^D = P(q) W_1^D$.

Let $n = 2g + 1$ or $n = 2g + 2$, $g \in \mathbb{N}$, be the degree of u^2 and let us consider the hyperelliptic surface of genus g, generated by (q, u) and associated to the constant energy surface $\mathcal{H} = \mathcal{E}$,

$$\mathcal{R}_{\mathcal{E}} = \left\{ (q, u) \in \mathbb{C}^2 : u^2 = A_0 \prod_{i=1}^{n} (q - z_i) \right\} \tag{6}$$

with $A_0 \in \mathbb{C}$ and $z_i \in \mathbb{C}$ being the roots of $u^2 = 0$ in (5). By construction, u represents physically a velocity, since

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p} = V_0(q) p = u$$

and so, $\mathcal{R}_{\mathcal{E}}$ is the affine part of the hyperelliptic surface of genus g (see [6, 13, 21]) associated to $\mathcal{H} = \mathcal{E}$ in the configuration space variables (q, u). Except that for a finite number of singular energy points, the roots z_j of u^2 are all simple. Such finite set corresponds to separatrices which, in the following, will be called singular curves. After adding the points at infinity, $\mathcal{R}_{\mathcal{E}}$ is a compact non-degenerate Riemann surface which can be represented as a two-sheeted branched cover of the Riemann sphere $\mathbb{C}$ (see [21] for instance). In the following we use the same symbol $\mathcal{R}_{\mathcal{E}}$ also for the compact Riemann surface associated to (5).

To any γ closed cycle in $\mathcal{R}_{\mathcal{E}}$, we may associate a local action variable just using formula (2) above

$$\mathcal{I}_\gamma(\mathcal{E}) = \frac{1}{2\pi} \oint_\gamma p(\mathcal{E}, q) \, \mathrm{d}q \tag{7}$$

and the local angle variable

$$\phi = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}\mathcal{I}_\gamma} \int_{(q_0, u_0)}^{(\bar{q}, \bar{u})} \frac{\partial p}{\partial \mathcal{E}}(\mathcal{E}, q) \, \mathrm{d}q \tag{8}$$

where $(q_0, p_0) \in \gamma$ is fixed, $(\bar{q}, \bar{p}) \in \gamma$ and the integration path in (8) is entirely contained in γ. Let us notice that $\frac{\partial \mathcal{H}}{\partial \mathcal{I}_\gamma}$ is well defined since γ is fixed. The action-angle variables defined in (7) and (8) are the direct generalization of the usual action-angle variables associated to the real bounded motions of $\mathcal{H}$. In particular, (7) and (8) give action-angle variables associated to the real unbounded motions since γ may be chosen as the representative in the equivalence class corresponding to a real unbounded motion.

In the physically interesting examples, ϕ is an integral of the first kind except at the singular energy points, where $\mathrm{d}\phi$ becomes a differential of the third kind (an Abelian differential is a meromorphic $(1,0)$-form on a compact Riemann surface and is called of the first, second or third kind if it is holomorphic, has zero residues, does not satisfy an additional condition, respectively (see [21] or [13])).

The following proposition sets the conditions for $\frac{\partial p}{\partial \mathcal{E}}(q,u)\,\mathrm{d}q$ to be a holomorphic (1,0)-form except at the singular energy points.

Proposition 1. Let $H = \frac{1}{2}V_0 p^2 + V_1$ be a Hamiltonian with V_0 and V_1 as in (3). Let

$$\frac{\partial p}{\partial \mathcal{E}}(q,u)\,\mathrm{d}q = \frac{Q(q)}{u}\,\mathrm{d}q$$

with Q and u as in (5). Then the roots of $u^2 = 0$ are all simple except at a finite number of isolated singular energy points.

If $\deg Q \leqslant g-1$, then $\mathrm{d}\phi$ is a holomorphic $(1,0)$ form on $\mathcal{R}_{\mathcal{E}}$ except on a finite number of singular curves.

The proof is a direct consequence of the form (1) of the Hamiltonian, of the definition of the angle variable and of the requirements on Q and u.

In the following, we consider Hamiltonians fulfilling proposition 1. We call the angle defined in (8) local since the Abelian integral cannot be inverted, in general, if $g \geqslant 2$. Indeed, as it is well known, ϕ may take on arbitrary values if we change the integration path keeping the endpoints fixed. In the next section, we use Jacobi inversion theorem and introduce a vector of angles globally well defined.

We call $\mathcal{I}_\gamma$ an action of the first kind if it is the combination of complete Abelian intergrals of the first kind; of the second kind if it is the combination of complete Abelian integrals of the first and second kind; of the third kind in all other cases. In general the action variable is a complete integral of the second kind. The following proposition accounts for the remaining cases, which we consider here for completeness and for later use in examples.

Proposition 2. (a) Let $\mathcal{H}$ be as in proposition 1 and let $\deg u^2 = 2g+1$. Then $p\mathrm{d}q$ is an integral of the third kind if and only if P has simple roots or $\sqrt{\frac{V_0^D}{W_1^D}}$ is a rational function with a nontrivial denominator.

(b) Let $\mathcal{H}$ be as in proposition 1 and $\deg u^2 = 2g+1$ or $2g+2$ and $P \equiv$ constant. Let $V_0^D = W_1^D T^2$, where T^2 is a polynomial with all roots of even multiplicity and

$$\begin{aligned} &\deg V_1^N + \deg T \leqslant g-1 \\ &\tfrac{1}{2}\deg V_1^D + \tfrac{1}{2}\deg V_0^D \leqslant g-1. \end{aligned} \tag{9}$$

Then, $\mathcal{I}_\gamma$ is a complete integral of the first kind.

Sketch of the proof. Let us observe that the hypothesis $\deg u^2$ odd in proposition 2(a) is not restrictive up to a degree one rational transformation.

The proof of the above proposition is straightforward and amounts to a computation using the following formulae

$$p\,\mathrm{d}q = \frac{2(\mathcal{E}-V_1)Q}{u}\,\mathrm{d}q. \tag{10}$$

Since, $Q = \sqrt{V_0^D W_1^D}$ and $V_1^D = P W_1^D$, then

$$\frac{Q}{V_1^D} = \frac{1}{P(q)} \sqrt{\frac{V_0^D}{W_1^D}}. \tag{11}$$

□

Remark 1. Let $p\mathrm{d}q$ be a differential of the third kind, then, according to (11), the simple poles of $\mathcal{I}_\gamma$ do not occur among the roots (finite or infinite) of $u^2 = 0$. Let $\{(a_i, b_i)\}$ be the finite set of its simple poles on $\mathcal{R}_\mathcal{E}$, then γ in (7) is in $\mathcal{R}_\mathcal{E}{}'$, the Riemann surface $\mathcal{R}_\mathcal{E}$ except for a regular pathwise-connected cut which passes once through each pole (a_i, b_i).

The main proposition of this section is the following one since it gives the dependence of $\mathcal{I}_\gamma$ on γ and on z_i's the roots of $u^2 = 0$. As a consequence we also get the dependence of $\mathcal{I}_\gamma$ on $\mathcal{E}$. The main application of the following proposition, however, will be in next section where we characterize the monodromy properties of the action variable in function of $\mathcal{E}$.

Proposition 3. Let $\mathcal{H}$ be as in proposition 1 and $\deg u^2 = 2g + 1$ or $2g + 2$. Let $w_j(\gamma) = \int_\gamma \mathrm{d}t_j$, $j = 1, \dots, g$, be a basis of integrals of the first kind computed along the path γ. Let $\bar{\omega}_{(c,d)}^{(a,b)}(\gamma)$ be the integral of the third kind associated to the logarithmic singular points (a, b), $(c, d) \in \mathcal{R}_\mathcal{E}$ with residue normalized to $+1$ and -1 respectively. Let $\zeta_{(a,b)}^{\nu}(\gamma)$ be the integral of the second kind with a single pole of order ν at the point $(a, b) \in \mathcal{R}_\mathcal{E}$ and residue normalized to 1. Then the action computed along the cycle γ can be expressed as

$$\begin{aligned} \mathcal{I}_\gamma = \frac{1}{2\pi} \oint_\gamma p\, \mathrm{d}q = \sum_{i=1}^{\bar{i}} R_i \bar{\omega}_{(c_i,d_i)}^{(a_i,b_i)}(\gamma) + \sum_{l=1}^{g} \lambda_l w_l(\gamma) \\ + \sum_{(e_j, f_j)} [A_1 \zeta_{(e_j,f_j)}^{(1)}(\gamma) + \cdots + A_\nu \zeta_{(e_j,f_j)}^{(\nu)}(\gamma)] \end{aligned} \tag{12}$$

where $\sum_{i=1}^{\bar{i}} R_i = 0$, (a_i, b_i), (c_i, d_i), $i = 1, \dots, \bar{i}$ are the simple poles of $p\mathrm{d}q$ and (e_j, f_j) are poles of order ν of $p\mathrm{d}q$.

Moreover, the coefficients R_i, A_ν, λ_l are polynomials which are symmetric in the roots z_j of $u^2 = 0$ and invariant under permutations of the indices of the z_i's.

Sketch of the proof. The proof of proposition 3 is a direct consequence of the fact that the action is expressed as the sum of complete integrals of the first, second and third kind. The coefficients R_i, A_ν and λ_l are symmetric polynomials in the roots of $u^2 = 0$ by construction due to the dependence of p on (q, u).

The property of invariance follows immediately from the following considerations. By linearity of the integral, $\mathcal{I}_{n\gamma+\gamma'} = n\mathcal{I}_\gamma + \mathcal{I}_\gamma$, where $n\gamma + \gamma'$, $n \in \mathbb{Z}$, denotes any path in the corresponding equivalence class. As is well known, any unimodular transformation of the period matrix induces a change of basis of cycles on $\mathcal{R}_\mathcal{E}$ and a well-defined permutation of the roots z_i's. Then, since equality must still hold in (12) after a unimodular transformation, the coefficients R_i, A_ν and λ_l cannot be affected by it, otherwise a contradiction occurs. This can only occur if the coefficients are invariant under the permutation of roots z_i's. □

In the following section, we show that closed paths in the energy plane are in one-to-one correspondence with unimodular transformations and, then, we completely characterize the analytical properties of $\mathcal{I}_\gamma$ as a function of $\mathcal{E}$.

3. Global action-angle variables

We consider here the dependence of $\mathcal{I}_\gamma$ on $\mathcal{E} \in \mathbb{C}$ and we propose a globally well-posed definition of the angle variable on $\mathcal{R}_\mathcal{E}$. We introduce a complex vector of angles and associate to it the corresponding complex matrix of actions. The main consequence of this definition is that the monodromy properties of the action matrix are expressed in an elegant and closed form in theorem 4.

The situation concerning the local angle variable is more delicate. Indeed, changing wisely the integration path, ϕ may assume any value if $g > 1$. Anyway, from proposition 1 above, $\mathrm{d}\phi$ is a holomorphic (1,0)-form. Since there exist g independent such forms, we may, formally, associate to each Abelian differential of the first kind the corresponding angle variable on $\mathcal{R}_\mathcal{E}$. The name local angle variables is meaningful, since each new variable evolves linearly in a suitable rescaled time variable, as required for real angle variables. Indeed, the passage from one local angle to another corresponds to a nonlinear time rescaling transformation, which preserves the Hamilton equations, but not the symplectic structure of the phase space. Then the global angle variables are defined as follows. To each g-tuple of independent $(1, 0)$-forms, there is associated a g-tuple of angle variables on which we may apply Jacobi inversion theorem to obtain q and p as functions of the angle variables.

Each differential of the first kind $\mathrm{d}t_j$ is a 'time' differential in our set-up. Let $(q, p) \to (q, p_j)$ be the change of variables such that

$$\mathcal{H}(q, p) = \mathcal{H}_j(q, p_j) = \tfrac{1}{2} V_0^{(j)}(q) p_j^2 + V_1^{(j)}(q) \tag{13}$$

and the local angle variable with respect to (q, p_j) is

$$\frac{\partial p_j}{\partial \mathcal{E}}(q, \mathcal{E})\,\mathrm{d}q = \mathrm{d}t_j. \tag{14}$$

With this construction, we get g independent local angle variables and the inverse transformation from action-angle variables to the original (q, p) variables is well defined globally. Let $a \in \mathbb{C}$ fixed, then a basis of holomorphic $(1, 0)$-forms is given by

$$\mathrm{d}t_j = \frac{(q-a)^{j-1}}{u}\mathrm{d}q \qquad j = 1, \ldots, g \tag{15}$$

that is, there exists a unique g-tuple $\{c_j\}$ such that $\mathrm{d}t = \sum c_j\,\mathrm{d}t_j$. From (14) and (15) we get

$$\frac{(q-a)^{j-1}}{u} = \frac{Q}{u}\sqrt{\frac{V_0}{V_0^{(j)}}}$$

so that, in (13)

$$V_0^{(j)}(q) = \frac{Q^2(q)}{(q-a)^{2j-2}} V_0(q) \qquad V_1^{(j)}(q) \equiv V_1(q) \qquad \forall j = 1, \ldots, g. \tag{16}$$

Then $p_j = \frac{(q-a)^{j-1}}{q} p$ and the local action-angle variables with respect to the new time differential $\mathrm{d}t_j$ are

$$\begin{aligned}
\mathcal{I}_{j,\gamma} &= \frac{1}{2\pi}\oint_\gamma p_j\,\mathrm{d}q = \frac{1}{2\pi}\oint_\gamma \frac{\mathrm{d}t_j}{\mathrm{d}t} p\,\mathrm{d}q = \oint_\gamma \frac{(\mathcal{E}-V_1)}{\pi u}(q-a)^{j-1}\,\mathrm{d}q \\
\phi_j &= \frac{\partial\mathcal{H}}{\partial\mathcal{I}_{j,\gamma}}\int_{(q_0,u_0)}^{(\bar q,\bar u)} \frac{\partial p_j}{\partial\mathcal{E}}\,\mathrm{d}q = \frac{\partial\mathcal{H}}{\partial\mathcal{I}_{j,\gamma}}\int_{(q_0,u_0)}^{(\bar q,\bar u)} \frac{\mathrm{d}p_j}{\mathrm{d}p}\frac{\partial p}{\partial\mathcal{E}}\,\mathrm{d}q = \frac{\partial\mathcal{H}}{\partial\mathcal{I}_{j,\gamma}}\int_{(q_0,u_0)}^{(\bar q,\bar u)} \frac{\mathrm{d}t_j}{\mathrm{d}t}\,\mathrm{d}t \\
&= \frac{\partial\mathcal{H}}{\partial\mathcal{I}^{j,\gamma}}\int_{(q_0,u_0)}^{(\bar q,\bar u)} \mathrm{d}t_j.
\end{aligned} \tag{17}$$

Let us notice that the symplectic structure of the phase space after the change of variables is no more canonical in the old variables (q, p).

Let $\mathcal{H}$ be as in proposition 1, let $\mathrm{d}t \equiv \mathrm{d}t_1, \ldots, \mathrm{d}t_g$ be a basis of holomorphic $(1,0)$-forms and let $\gamma \equiv \gamma_1, \ldots, \gamma_{2g}$ be a canonical basis of cycles on $\mathcal{R}_{\mathcal{E}}$ or $\mathcal{R}_{\mathcal{E}}'$ according to the remark after proposition 2. The matrix of actions is defined by

$$\mathcal{I}_{j,i} = \frac{1}{2\pi} \oint_{\gamma_i} p_j \, \mathrm{d}q \qquad j = 1, \ldots, g \qquad i = 1, \ldots, 2g. \tag{18}$$

It is made of integrals of the second kind, except when proposition 2 applies. In particular, it is not restrictive to suppose that all the integrals are of the same kind (that is of the first/second or of the third kind) for $j = 1, \ldots, g$. Each action may be expressed as an appropriate finite sum as in (12), where the basis of cycles is in $\mathcal{R}_{\mathcal{E}}$ if the action matrix is of the second or first kind, in $\mathcal{R}_{\mathcal{E}}'$ is the action matrix is of the third kind.

By proposition 3, the coefficients R_i, A_ν, λ_l are symmetric polynomials in the roots of $u^2 = 0$ invariant under permutations of such roots. Then the same unimodular transformation of the period matrix and of the action matrix is associated to a permutation of such roots. Let us show that to each singular point $\mathcal{E}$ there is associated a well-defined unimodular transformation of the action matrix and that the monodromy properties of $\mathcal{I}_{j,i}$ in function of $\mathcal{E}$ are given by subgroups of unimodular transformations. As a consequence, we classify the singular curves and prove the analytical properties of the complex action with respect to $\mathcal{E}$.

Let us denote with $\mathcal{U}_{\mathcal{E}}$ the set of singular energy points and with $\mathcal{G}_{\mathcal{E}}$ the set of transformations associated to $\mathcal{U}_{\mathcal{E}}$ in the following way. To each singular energy point $\mathcal{E}$ we associate two transformations (a matrix and its inverse), which correspond to the change of the matrix of actions after a complete turn around $\mathcal{E} \in \mathcal{U}_{\mathcal{E}}$ clockwise or anticlockwise, respectively, excluding all other singular energy points.

Since a turn around a singular energy point corresponds to a well-defined root exchange of $u^2 = 0$ and permutations of roots produce unimodular transformations of the period matrix, by proposition 3, we conclude that the elements of $\mathcal{G}_{\mathcal{E}}$ are unimodular transformations.

Clearly, $\mathcal{G}_{\mathcal{E}}$ generates a subgroup of unimodular transformations which characterizes completely the analyticity properties of $\mathcal{I}_\gamma$ in function of $\mathcal{E}$. Indeed, to each closed cycle in the complex energy plane there is associated a uniquely defined unimodular transformation given by a convenient composition of elements in $\mathcal{G}_{\mathcal{E}}$. The converse is also true: to any product of matrices in $\mathcal{G}_{\mathcal{E}}$ there corresponds a well-defined equivalence class of closed cycles in the complex energy plane. In particular, the identity matrix corresponds to the equivalence class of paths which avoid all singular points.

Indeed we have just proven the following.

Theorem 4. Let $\mathcal{H}$ be as in proposition 1 and $\mathcal{I} = \{\mathcal{I}_{j,i}\}_{j=1,g}^{i=1,2g}$ be the action matrix defined in (18). Let $\mathcal{U}_{\mathcal{E}}$ be the set of singular energy points associated to $\mathcal{H}$. Then a well-defined unimodular transformation of the action matrix and its inverse is associated to each element of $\mathcal{U}_{\mathcal{E}}$. The set of these transformations, $\mathcal{G}_{\mathcal{E}}$ generates a subgroup of unimodular transformations in terms of which the monodromy of $\mathcal{I}$ in function of $\mathcal{E}$ is completely determined.

As a consequence, the action matrix is an analytic function of $\mathcal{E}$ which may be analytically continued but is not single valued on $\mathbb{C} - \mathcal{U}_{\mathcal{E}}$.

In the following and in the last section, we will consider some examples and compute the unimodular transformations associated with them.

We close this section by defining the complex angle variable as an angle vector $\bar{\psi}(q) = (\psi_1(q), \ldots, \psi_g(q))$ where

$$\psi_j(q) \equiv \psi_{j,i}(q) = \frac{\partial \mathcal{H}_j}{\partial \mathcal{I}_{j,i}} \int_{(q_0,u_0)}^{(q,u)} \mathrm{d}t_j \qquad j = 1, \ldots, g \tag{19}$$

where $i \in \{1, \ldots, 2g\}$ is fixed. As usual (q_0, p_0) is fixed in $\mathcal{R}_{\mathcal{E}}$. From Jacobi inversion theorem, the complex angle variable is well defined globally on the complex Abelian torus of dimension g associated to $\mathcal{R}_{\mathcal{E}}$ and symmetric functions of q are meromorphic functions in the angles ψ_j's. In this sense we may invert equation (19). Moreover, each angle variable ψ_j defines a local angle variable in the sense specified in section 2 which evolves linearly with respect to the timescale $\mathrm{d}t_j$.

Let us notice that, the coefficient $\frac{\partial \mathcal{H}}{\partial \mathcal{I}_{j,i}}$ is well defined since we have fixed an element γ in the set of closed cycles.

We call $(\mathcal{I}_{i,j}(\mathcal{E}), \psi_j(q))$, $j = 1, \ldots, g$, $i = 1, \ldots, 2g$, global action-angle variables. From the above discussion it is clear that we may give a global complex characterization of the Hamiltonian system using them.

4. Examples

Example 1. Let

$$\mathcal{H}(p, q) = \frac{1}{2}p^2 + \Omega \frac{q^2}{2} - \frac{q^3}{3}$$

with $\Omega \in \mathbb{R}^+$. Then the generic energy surface has genus 1; moreover in this case $u = p$. The singular energy points are $\mathcal{E} = 0, \frac{\Omega^3}{6}$ and become coincident for $\Omega = 0$.

Since $g = 1$, for each regular energy, there is one differential of the first kind $\mathrm{d}t = \frac{\mathrm{d}q}{p(q,\mathcal{E})}$ and the canonical basis of cycles is $\gamma_1 = [z_2, z_3]$, $\gamma_2 = [z_1, z_2]$, where z_i, $i = 1, 2, 3$ are the roots of $p^2 = 0$. With the notation $[z_i, z_j]$ we mean a cycle which turns around z_i, z_j once and does not include the other roots and ∞ point.

If $\Omega > 0$ and $\mathcal{E} \in]0, \frac{\Omega^3}{6}[$, z_i's are real and we order them as follows $z_1 < z_2 < z_3$. The action vector is then

$$\mathcal{I}_{\gamma_1} = \frac{1}{2\pi} \oint_{\gamma_1} p\mathrm{d}q = \frac{2\mathrm{i}\sqrt{2}}{15\pi\sqrt{3}} a_{13}^{\frac{5}{2}} \{-(k')^2(2 - k^2)\mathfrak{K}(k) + 2(1 - k^2 + k^4)\mathfrak{E}(k)\}$$

$$\mathcal{I}_{\gamma_2} = \frac{1}{2\pi} \oint_{\gamma_2} p\mathrm{d}q = \frac{2\sqrt{2}}{15\pi\sqrt{3}} a_{13}^{\frac{5}{2}} \{-k^2(1 + k^2)\mathfrak{K}'(k) + 2(1 - k^2 + k^4)\mathfrak{E}'(k)\}$$

where

$$\mathfrak{K}(k) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \qquad \mathfrak{K}'(k) = \mathfrak{K}(k')$$

$$\mathfrak{E}(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi}\,\mathrm{d}\phi \qquad \mathfrak{E}'(k) = \mathfrak{E}(k')$$

$$a_{ij} = z_i - z_j \qquad 0 < k^2 = \frac{z_2 - z_3}{z_1 - z_3} < 1$$

$$\omega_1 = \int_{z_1}^{z_2} \frac{\mathrm{d}q}{p} = \frac{1}{\sqrt{a_{13}}}\mathfrak{K}(k) \qquad \omega_2 = \int_{z_2}^{z_3} \frac{\mathrm{d}q}{p} = \frac{1}{\sqrt{-a_{13}}}\mathfrak{K}'(k)$$

where ω_1 and ω_2 are the fundamental semiperiods associated to $\mathcal{R}_{\mathcal{E}}$. Clearly $\mathcal{I}_{\gamma_1}(k) = \sqrt{-1}\mathcal{I}_{\gamma_2}(k')$.

First of all, the local action variable of the real bounded or unbounded motion in the range $\mathcal{E} \in]0, \frac{\Omega^3}{6}[$ are the same: $\mathcal{I}_{\gamma_2}$. Indeed, $[z_2 z_1]$ is equivalent to $[z_3 \infty]$ on $\mathcal{R}_{\mathcal{E}}$.

We now describe the monodromy properties of the action vector with respect to to $\mathcal{E} \in \mathbb{C}$, when $\Omega > 0$. As a consequence, we show the existence of the limiting action vector when $\mathcal{E} \to 0$ or $\mathcal{E} \to \Omega^3/6$, $\mathcal{E} \in \mathbb{C}$.

To start with, we restrict ourselves to real energies and take the limit $\mathcal{E} \to 0$ ($k^2 \to 1$) and observe that, from the well known formulae $\mathfrak{E}(1) = 1$, $\mathfrak{E}(0) = \pi/2$, $\mathfrak{K}(0) = \pi/2$ and $\lim_{k\to 0^+} \mathfrak{K}'(k) - \log(4/k) = 0$,

$$\lim_{\mathcal{E}\to 0^+} \mathcal{I}_{\gamma_1} = \frac{3}{5\pi}\Omega^2\sqrt{-\Omega} \qquad \lim_{\mathcal{E}\to 0^+} \mathcal{I}_{\gamma_2} = 0. \tag{20}$$

Let us now consider the analytical properties of $\mathcal{I}_\gamma$ with respect to $\mathcal{E} = 0$. Let $\mathcal{E} = \epsilon \exp(\mathrm{i}\theta)$, where $\theta \in [0, 2\pi]$. A complete turn around $\mathcal{E} = 0$ corresponds to an exchange between the roots z_1 and z_2. This induces the following unimodular transformation on the actions:

$$\mathcal{I}_{\gamma_1} \to \mathcal{I}_{\gamma_1} + \mathcal{I}_{\gamma_2} \qquad \mathcal{I}_{\gamma_2} \to \mathcal{I}_{\gamma_2}$$

which, actually, characterizes completely the nature of the singularity $\mathcal{E} = 0$ for $\mathcal{I}_\gamma$. Moreover, from the form of this transformation, we get that

$$\lim_{\mathcal{E}\to 0, \mathcal{E}\in\mathbb{C}} I_{\gamma_i} = \lim_{\mathcal{E}\to 0^+, \mathcal{E}\in\mathbb{R}} I_{\gamma_i} \qquad i = 1, 2$$

that is we define a meaningful complex action variable also in the degenerate case $\mathcal{E} = 0$ using (20).

In an analogous way, if $\mathcal{E} \to \Omega^3/6^-$($k^2 \to 0$, real limit!), then

$$\lim_{\mathcal{E}\to\Omega^3/6^-} \mathcal{I}_{\gamma_1} = 0 \qquad \lim_{\mathcal{E}\to\Omega^3/6^-} \mathcal{I}_{\gamma_2} = \frac{3}{5\pi}\Omega^2\sqrt{\Omega}.$$

The exchange between the roots z_2 and z_3 induces the following unimodular transformation on the actions

$$\mathcal{I}_{\gamma_1} \to \mathcal{I}_{\gamma_1} \qquad \mathcal{I}_{\gamma_2} \to \mathcal{I}_{\gamma_1} + \mathcal{I}_{\gamma_2}$$

and, as before, the complex action is well defined in the limiting case $\mathcal{E} \to \Omega^3/6$, $\mathcal{E} \in \mathbb{C}$.

In general, to any possible combination of turns around the two singular energy points, there corresponds a well-defined unimodular transformation of the actions given by the corresponding composition of the generating unimodular transformations

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{21}$$

and their inverses. Since (21) generate the entire group of 2×2 unimodular transformations, by theorem 4, an equivalence class of paths in the complex energy plane corresponds to any unimodular transformation.

We now briefly consider the case $\Omega = 0$ along the same lines as above. In this case $\mathcal{I}_{\gamma_1}, \mathcal{I}_{\gamma_2} \approx \mathcal{E}^{\frac{1}{6}}$, as $\mathcal{E} \to 0$ and to any turn around the singular energy point $\mathcal{E} = 0$ there corresponds a cyclic permutation of the roots z_i. In particular,

$$\mathcal{I}_{\gamma_1} \to \mathcal{I}_{\gamma_2} \qquad \mathcal{I}_{\gamma_2} \to -\mathcal{I}_{\gamma_1} - \mathcal{I}_{\gamma_2}.$$

After three turns around $\mathcal{E} = 0$ we obtain the identity transformation, and the set of possible paths around the energy singularity is associated to the proper subgroup of unimodular transformations generated by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Example 2. Let us consider a subcase of (1)

$$\mathcal{H}(q,p)=\tfrac{1}{2}p^2+V(q) \tag{22}$$

where $V(q)$ is a polynomial of degree $n=2g+1\geqslant 5$. Then the generic $\mathcal{R}_{\mathcal{E}}$ has genus $g\geqslant 2$. We choose the following basis of differentials of the first kind

$$\mathrm{d}t_j=\frac{q^{j-1}}{p}\mathrm{d}q \qquad j=1,\ldots,g$$

and we fix a canonical basis of cycles on $\mathcal{R}_{\mathcal{E}}$, $\gamma_1,\ldots,\gamma_{2g}$.

The action matrix and the angle vector are given by

$$\mathcal{I}_{j,i}=\frac{1}{2\pi}\oint_{\gamma_i}q^{j-1}p\,\mathrm{d}q \qquad j=1,\ldots,g \qquad i=1,\ldots,2g$$
$$\psi_j(\bar{q})=\frac{\partial\mathcal{H}}{\partial\mathcal{I}_{j,i}}\int_{(q_0,u_0)}^{(\bar{q},\bar{u})}\frac{q^{j-1}}{p}\,\mathrm{d}q \qquad j=1,\ldots,g.$$

In this case the action matrix is a complete hyperelliptic integral of the second kind with poles at ∞. Indeed, let $\zeta^{(\nu)}(q,p;\infty)$ be the integral which becomes infinite only at the point ∞ with principal part $z^{\frac{\nu+1}{2}}$, where z is the local coordinate at the infinity point ($q=z^{-2}$). In our case, $\nu=2,4,\ldots,2(j+g+1)$, and with the notation $\nu=2\mu$,

$$\zeta^{(2\mu)}(q,p;\infty)=\frac{2\mu+1}{2}\int_{(q_0,u_0)}^{(q,u)}\frac{q^{g+\mu}Q_\mu}{p}\,\mathrm{d}q \qquad \mu=1,\ldots,j+g+1$$

where $p=q^{g+\frac{1}{2}}w$, and w may be developed as $w=a_0+\frac{a_1}{q}+\frac{a_2}{q^2}+\cdots$ and Q_μ is a degree μ polynomial in $\frac{1}{q}$ given by $Q_\mu(q)=a_0+\frac{a_1}{q}+\cdots+\frac{a_\mu}{q^\mu}$. In particular we may express the actions as:

$$\mathcal{I}_{j,i}=\sum_{\mu=1}^{g+1+j}\delta_\mu^{(j)}\zeta^{2\mu}(\gamma_i)+\sum_{l=1}^{g}\eta_l^{(j)}\tau_l(\gamma_i)$$

where $\tau_j(\gamma_i)=\oint_{\gamma_i}\mathrm{d}t_j$. $\delta_\mu^{(j)}$, $j=1,\ldots,g$, $\mu=1,\ldots,g+j+1$ are degree $\mu-1$ homogeneous polynomials in the roots of p^2 invariant under permutations. $\eta_l^{(j)}$, $j=1,\ldots,g$, $l=1,\ldots,g$ are degree $g+j+l$ homogeneous polynomials in the roots of p^2 invariant under permutations of the variables.

The monodromy properties listed in the previous section for the action matrix may then be calculated.

We end this example writing down the Hamiltonian expressed with respect to the time scalings associated to the time differentials $\mathrm{d}t_j$. With our choice of the base of differentials of the first kind, $p_j=q^jp$, $j=1,\ldots,g$,

$$\mathcal{H}_j(q,p_j)=\frac{p_j^2}{2q^{2j}}+V(q).$$

Example 3. Let $a_1,\ldots,a_5$ be real and such that $a_j\neq 1$, $j=1,\ldots,5$; then the Riemann surface $\mathcal{R}_{\mathcal{E}}$ associated to $u^2=2(\mathcal{E}(q-1)^2-\prod_{i=1}^5(q-a_i))$ has genus $g=2$ for almost any $\mathcal{E}$. Let us consider the associated Hamiltonian

$$\mathcal{H}(q,p)=\frac{p^2}{2}+\frac{1}{(q-1)^2}\prod_{i=1}^{5}(q-a_i). \tag{23}$$

In this case, the local action integral is of the third kind and the local action-angle variables are

$$\mathcal{I}_\gamma = \frac{1}{2\pi}\oint_\gamma p\mathrm{d}q = \oint_\gamma \mathrm{d}q \frac{(\mathcal{E}(q-1)^2 - \prod(q-a_i))}{\pi u(q-1)}$$
$$\phi = \frac{\partial \mathcal{H}}{\partial \mathcal{I}_\gamma}\int_{(q_0,u_0)}^{(q,u)} \mathrm{d}q \frac{q-1}{u}.$$

Let us choose as basis of differentials of the first kind $\mathrm{d}t_1 = \frac{1}{u}\mathrm{d}q$, $\mathrm{d}t_2 = \frac{q}{u}\mathrm{d}q$. Let $\gamma_1, \ldots, \gamma_4$ be a basis of cycles on $\mathcal{R}_\mathcal{E}'$, obtained cutting $\mathcal{R}_\mathcal{E}$ along a path connecting the points $(1, \pm\sqrt{-\prod a_i})$. Then, in the coordinates (q, p_j), $j = 1, 2$ the Hamiltonian becomes

$$\mathcal{H}_j(q, p) = \frac{(q-1)^2}{2q^{2j-2}} p^2 + \frac{1}{(q-1)^2}\prod_{i=1}^{5}(q - a_i)$$

and the action matrix and the angle vector are the following

$$\mathcal{I}_{j,i} = \frac{1}{2\pi}\oint_{\gamma_i} p_j\, \mathrm{d}q = \oint_{\gamma_i} \mathrm{d}q\, \frac{\mathcal{E}(q-1)^2 - \prod_{k=1}^5(q-a_k)}{\pi(q-1)^2 u} \qquad i = 1, \ldots, 4 \qquad j = 1, 2$$
$$\psi_j(\bar{q}) = \frac{\partial \mathcal{H}}{\partial \mathcal{I}_{j,i}}\int_{(q_0,u_0)}^{(\bar{q},\bar{u})} \frac{q^j}{u}\mathrm{d}q \qquad j = 1, 2.$$

The action matrix is then of the third type with two simple poles on $\mathcal{R}_\mathcal{E}$ in $(1, \pm\prod_{k=1}^5(1 - a_k))$, and poles at infinity with principal part $z^{\frac{\nu+1}{2}}$ where $\nu = 2, 4$.

5. The case of reducible Hamiltonians

In this section we consider the peculiar class of Hamiltonians for which a local angle variable ϕ is reducible to an elliptic integral of the first kind through a rational transformation of q of degree η, where $\eta \geqslant 2$. In this case, ϕ is a globally well-defined function of q. Indeed it takes a finite number, η, of values, as we arbitrarily change the integration path in (5), and the inverse function $q(\phi)$ is given by the composition of a rational function of degree η with a meromorphic function.

Such reducible cases represent the direct generalization of the elliptic case for systems with one degree of freedom and energy surfaces of genus $g > 1$. In this case, local and global action variables coincide as for the case $g = 1$.

We show that the change of phase coordinates induced by the rational transformation is canonical and the symplectic structure is conserved. This observation has important consequences for the monodromy properties of such reducible cases. Indeed, suppose we have a family of reducible Hamiltonians depending analytically on a parameter (e.g. $\mathcal{E}$), then the monodromy of the reducible action vector may be described in terms of that of the (reduced) elliptic action integral.

We start recalling two classical theorems which set up the reducible case, then we consider the properties of the action-angle variables related to the reducible hyperelliptic integral.

Theorem A (Briot–Bouquet) [19, 6, 17]. Let

$$\mathcal{F}(\dot{q}, q) = 0 \tag{24}$$

be an algebraic equation of two variables of degree m in $\dot{q}$. Suppose that, around the movable critical points, the equation takes η values, then only one of the following three possibilities is verified: q is an algebraic function of t, or q is an algebraic function of $\exp(gt)$, or q is an algebraic function of $\mathrm{sn}_{k^2}(gt)$, where g and k are constant.

Let us denote $f(q) = 1/\dot{q}$, where $\dot{q}$ is the algebraic function of q defined by (24). Then the case we consider here corresponds to the third possibility, when the independent periods τ of the Abelian integral $t = \int f(q)\,\mathrm{d}q = \mathcal{J}$ reduce to 2.

The equality

$$\mathrm{d}t = \frac{\mathrm{d}u}{g\sqrt{(1-u^2)(1-k^2u^2)}} = f(q)\,\mathrm{d}q$$

shows that the Abelian integral of the first kind $\mathcal{J}(q)$ is the algebraic transform of

$$\int \frac{\mathrm{d}u}{g\sqrt{(1-u^2)(1-k^2u^2)}}.$$

In our set-up ($\mathcal{H}$ as in proposition 1), if $\mathcal{E} \in \mathbb{C}$ is regular, $\mathcal{J}(q)$ is an Abelian integral of the first kind. So, in order to check that the angle integral admits a finite unknown number η of values, we have to show whether the corresponding Riemann surface $\mathcal{R}_{\mathcal{E}}$ of genus $g > 1$ is the rational transform of a Riemann surface of genus 1. Such reducible Riemann surfaces satisfy

Theorem B (Poincarè–Weierstrass [6, 21, 22]). If there exists a system of g Abelian integrals of rank g, among which there is one that may be reduced to an elliptic integral, and if we consider the corresponding theta function Θ, then:

(1) such a theta function with g variables is equivalent through a rational transformation of degree η to the product of a theta function of one variable and of a theta function of $(g-1)$ variables.

(2) With a linear transformation the theta function Θ may be changed to a form in which the period matrix has the following form:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \tau_{11} & \tau_{12} & \dots & \tau_{1g} \\ 0 & 1 & \dots & 0 & \tau_{21} & \tau_{22} & \dots & \tau_{2g} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \tau_{g1} & \tau_{g2} & \dots & \tau_{gg} \end{pmatrix}$$

where, as usual $\tau_{ij} = \tau_{ji}$, and period τ_{12} is commensurable to one, while the periods $\tau_{13}, \dots, \tau_{1g}$ are all zero.

(3) In particular, let $g = 2$. Then, if there exists an integral of the first kind corresponding to the algebraic relation

$$p^2 = q(1-q)(1-k^2q)(1-l^2q)(1-m^2q)$$

which has only two periods, it is possible to find a system of normal integrals whose period matrix is

$$\begin{pmatrix} 0 & 1 & G & \frac{1}{\eta} \\ 1 & 0 & \frac{1}{\eta} & G' \end{pmatrix}$$

where η is a positive integer.

Moreover, under such conditions, there exists a second integral of the first kind independent from the first and which enjoys the same properties.

In the following we call reducible a Hamiltonian which satisfies the conditions of proposition 1 and such that the associated local angle variable is a reducible hyperelliptic integral of the first kind. Then, there exists a degree η rational transformation

$$x = R_\eta(q) \tag{25}$$

such that

$$\int_{(q_0,u_0)}^{(q,u)} \frac{Q}{u}\mathrm{d}q = \int_{(x_0,\rho_0)}^{(x,\rho)} \frac{\mathrm{d}x}{y} \tag{26}$$

where, by construction $\frac{\mathrm{d}x}{y}$ is an elliptic differential of the first kind.

We now express the reducible Hamiltonian in (x, ρ), the conjugate coordinates associated to (25), and show that such transformation is canonical. Indeed

$$\mathcal{H}(q, p) = \bar{\mathcal{H}}(x, \rho) = \tfrac{1}{2}U_0\rho^2 + U_1 \tag{27}$$

where ρ satisfies the following compatibility condition

$$\frac{\partial\rho}{\partial\mathcal{E}} = \frac{1}{y}. \tag{28}$$

From (25), (27) and (28), we get that

$$U_0(x) = V_0(q)\left(\frac{\mathrm{d}R_\eta}{\mathrm{d}q}(q)\right)^2 \qquad U_1(x) = V_1(q). \tag{29}$$

Since $x(q)$ is rational of degree η there exist η distinct inverse determinations. We fix one of them and denote it $q(x)$, With this convention, we may invert (29) and express $\mathcal{H}$ in the new coordinates.

Theorem 5. Let $\mathcal{H}$ be a reducible Hamiltonian and R_η be the rational transformation between q and x. Then to R_η there corresponds a canonical change of coordinates $(q, p) \to (x, \rho)$.

Proof. It is sufficient to show that the local action integral of the old Hamiltonian is transformed into the global action integral of the new Hamiltonian

$$2\pi\mathcal{I}_\gamma = \oint_\gamma p\mathrm{d}q = \oint_\gamma \sqrt{\frac{2(\mathcal{E}-V_1)}{V_0(R'_\eta)^2}(R'_\eta)^2\mathrm{d}q} = \oint_{\bar{\gamma}} \sqrt{\frac{2(\mathcal{E}-U_1)}{U_0}}\,\mathrm{d}x = \oint_{\bar{\gamma}} \rho\,\mathrm{d}x = 2\pi\tilde{\mathcal{I}}_{\bar{\gamma}} \tag{30}$$

where $\bar{\gamma}$ is the cycle on the elliptic surface associated to $\bar{\mathcal{H}} = \mathcal{E}$ corresponding to γ. $\square$

Equation (30) characterizes completely the analytic properties of the reducible action vector when we have a family of reducible Hamiltonians. Indeed a straightforward consequence of theorem 5 is the following.

Corollary 6. Let $\mathcal{H}(\alpha)$ be a family of Hamiltonians depending analytically on the complex parameter α and let $\mathcal{H}(\alpha)$ be reducible in some complex domain $\alpha \in \mathcal{V}_\alpha$. Then the monodromy properties of $\mathcal{I}_\alpha$ and of $\tilde{\mathcal{I}}_\alpha$ are the same with respect to the singular points $\alpha \in \mathcal{V}_\alpha$.

In particular if $\mathcal{H}$ is reducible in same domain of $\mathcal{E}$, we may equivalently study the monodromy properties of $\mathcal{I}_\gamma$ directly or first consider the analytic behaviour of the reduced action $\tilde{\mathcal{I}}_{\gamma'}$ and then use equation (30) in order to get the monodromy of $\mathcal{I}_\gamma$. In the following section we apply corollary 6 to an example.

It seems natural to believe that non-reducible Hamiltonian could be obtained as limit of reducible ones, for instance, when $g = 2$ taking a convenient sequence of Hamiltonians $\mathcal{H}_\eta$ with $\eta \to \infty$ in (25). In this way, however, the limiting Hamiltonian will not satisfy proposition 1, since its period matrix is singular.

6. Examples of reducible Hamiltonians

We consider some examples when $g = 2$. In this case, from theorem B, it is always possible, in principle, to construct a reducible angle vector since there are two independent reducible hyperelliptic integrals of the first kind. In the case of the examples we consider below both integrals are explicitly known (see [6, 14] for other examples).

Example 4. The following integrals

$$\begin{aligned} \int \frac{\mathrm{d}q}{\sqrt{q^6 + Aq^4 + Bq^2 + C}} &= \int \frac{\mathrm{d}x}{2\sqrt{x(x^3 + Ax^2 + Bx + C)}} \\ \int \frac{q\,\mathrm{d}q}{\sqrt{q^6 + Aq^4 + Bq^2 + C}} &= \int \frac{\mathrm{d}x}{2\sqrt{x^3 + Ax^2 + Bx + C}} \end{aligned} \tag{31}$$

are equivalent under the degree $\eta = 2$ rational transformation

$$x = q^2. \tag{32}$$

Then

$$H = \frac{p^2}{2} + \frac{q^2}{2} - \frac{q^6}{6} \tag{33}$$

is reducible since its local angle variable is proportional to

$$\int \frac{\mathrm{d}q}{\sqrt{2\mathcal{E} - q^2 + \frac{1}{3}q^6}}.$$

Equation (33), expressed in the new conjugate coordinates (x, ρ), becomes

$$H = 2x\rho^2 + \frac{x}{2} - \frac{x^3}{6} \tag{34}$$

and the relation between action-angle variables in (q, p) and (x, ρ) is

$$\mathcal{I}_{l,\gamma} = \frac{1}{2\pi} \oint_{z_l^{(s)}, z_l^{(e)}} p\,\mathrm{d}q = \frac{1}{2\pi} \oint_{(z_l^{(s)})^2, (z_l^{(e)})^2} \rho\,\mathrm{d}x$$

$$\phi_l = \frac{\partial \mathcal{H}}{\partial \mathcal{I}_{l,\gamma}^{(0)}} \int_{q_0}^{q} \frac{\mathrm{d}q}{\sqrt{2\mathcal{E} - q^2 + \frac{1}{3}q^6}} = \frac{\partial \mathcal{H}}{\partial \mathcal{I}_{l,\gamma}^{(0)}} \int_{(q_0)^2}^{q^2} \frac{\mathrm{d}x}{2\sqrt{2x(\mathcal{E} - \frac{x}{2} + \frac{x^3}{6})}}.$$

In the above formula we suppose that the cycle γ_l goes once around the roots $z_l^{(s)}, z_l^{(e)}$ of $p^2 = 0$. Moreover, the inverse transformation, from $(\phi, \mathcal{I}_\gamma)$ to (q, p) is well defined since q is the square root of a meromorphic function and so the local complex action angles are also globally well defined.

We consider now the analyticity properties of the action vector with respect to $\mathcal{E}$. The original system has three singular energy points $\mathcal{E} = 0, \pm\frac{1}{3}$. $p^2 = 0$, at $\mathcal{E} = \pm\frac{1}{3}$, has two couples of coincident roots; at $\mathcal{E} = 0$ one couple of coincident roots. Below we show that the singularity $\mathcal{E} = 0$ is eliminable.

In this example it is quite easy to compute the monodromy of $\mathcal{I}_\gamma$ both directly and using corollary 6. We use the following convention. Let us denote with a_i, $i = 1, 2, 3$ the roots of

$y^3 - 3y + 6\mathcal{E} = 0$. The roots of $p^2 = 0$ are then denoted e_i, $i = -3, -2, -1, 1, 2, 3$ with the convention $e_i = \mathrm{sgn}(i)\sqrt{a_{|i|}}$. The canonical basis of cycles is $\gamma_1 = [e_{-2}, e_{-1}]$, $\gamma_2 = [e_1, e_2]$, $\gamma_3 = [e_{-3}, e_{-2}]$ and $\gamma_4 = \gamma_3 + [e_{-1}, e_1]$. Then $\mathcal{I}_{j,\gamma_1} = -\mathcal{I}_{j,\gamma_2} = \alpha$ and $\mathcal{I}_{j,\gamma_2} = \mathcal{I}_{j,\gamma_3} = \beta$, $j = 1, 2$.

In the reduced system (34), there are four roots z_i, $i = 1, 4$ and we use the following conversion table: $z_1 = a_1$, $z_2 = 0$, $z_3 = a_2$ and $z_4 = a_3$. The basis of cycles is then $\tilde{\gamma}_1 = [z_2, z_3]$ and $\tilde{\gamma}_2 = [z_1, z_2]$. With these conventions it is straightforward to compute $\mathcal{I}_\gamma$ in function of $\tilde{\mathcal{I}}_{\tilde{\gamma}}$; for instance $\mathcal{I}_{\gamma_1} = \tilde{\mathcal{I}}_{[z_3, z_1]}$.

Let us start with $\mathcal{E} = \frac{1}{3}$. To a complete turn around this singular point, there corresponds an exchange between $e_{\pm 3}$ and $e_{\pm 2}$ which induces the following unimodular transformation on the action matrix

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

We may also compute directly the unimodular transformations induced on the reduced action $\tilde{\mathcal{I}}_{\gamma'}$. Indeed a complete turn around the singularity $\mathcal{E} = \frac{1}{3}$ induces an exchange between the roots z_3 and z_4 and the following unimodular transformation on $\tilde{\mathcal{I}}_{\gamma'}$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The effect on the reducible action matrix is the same as before, as it can be checked by direct substitution, since $\mathcal{I}_{\gamma_1} \to \beta - \alpha$ and $\mathcal{I}_{\gamma_3} \to \beta$.

In an analogous way we may compute the monodromy properties with respect to the singular energy point $\mathcal{E} = -\frac{1}{3}$, to which there corresponds the exchange between $e_{\pm 2}$ and $e_{\pm 1}$ and the following unimodular transformation of the actions

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

In the reduced system, we use a different convention and associate $z_1 = a_1$, $z_2 = a_2$, $z_3 = 0$ and $z_4 = a_3$. In this way we have an exchange between z_1 and z_2 and the following unimodular transformation for the reduced action $\tilde{\mathcal{I}}_{\gamma'}$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then as before, in both cases, $\mathcal{I}_{\gamma_1} \to -\alpha$ and $\mathcal{I}_{\gamma_3} \to \beta - \alpha$.

Finally we consider the case $\mathcal{E} = 0$. Then in the reducible system we have an exchange between e_1 and e_{-1}, so that we expect that $\mathcal{I}_{\gamma_i} \to \mathcal{I}_{\gamma_i}$. Indeed the unimodular transformation is

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

In view of the reduced system, it is convenient to express $\rho^2 = 0$ in Weierstrass normal form. Then $\mathcal{E} = 0$ is a double root and it is easy to check that it induces the identity transformation on the periods $\tilde{\mathcal{I}}_{\gamma'}$ as required. This is of course a trivial remark since $\mathcal{E} = 0$ is a regular point of the reduced system.

Example 5. A second family of reducible Hamiltonians may be obtained in the case $\eta = 3$, where

$$\begin{aligned} \int \frac{\mathrm{d}q}{\sqrt{3(q^3+aq+b)(q^3+cq^2+d)}} &= \int \frac{\mathrm{d}x}{\sqrt{x[4(3x-a)^3-27(b+cx)^2]}} \\ \int \frac{q\,\mathrm{d}q}{\sqrt{3(q^3+aq+b)(q^3+cq^2+d)}} &= \int \frac{\mathrm{d}x}{\sqrt{x[4(c+3bx)^3+27(1-ax)^2]}} \end{aligned} \tag{35}$$

using, respectively, the degree three rational transformations

$$\begin{aligned} x &= \frac{q^3+aq+b}{3q-d} \\ x &= \frac{q^3+cq^2+d}{aq^3-3bq^2} \end{aligned} \tag{36}$$

where the coefficients a, b, c, d must satisfy the compatibility condition $d = \frac{4}{3}[ac+3b]$.

Let us consider the following Hamiltonian

$$\mathcal{H}(q,p) = \frac{1}{2}(3q^3+12)(3q-4)p^2 - \frac{q^3+1}{6q-8}. \tag{37}$$

It is easy to check that this Hamiltonian is reducible only for $\mathcal{E} = 0$, for the choice of parameters $a = 0$, $b = 1$, $c = 0$ $d = 4$ in the first of (36) and (37). It is of course possible to compute the reduced Hamiltonian

$$\tilde{\mathcal{H}}(x,\rho) = 54(x^3-1)\rho^2 - \tfrac{1}{2}x$$

which is equivalent to (37) only in the case $\mathcal{E} = 0$. The angle integral is then proportional to the first of (35) and the action integral is

$$\mathcal{I}_l = \frac{\mathrm{i}}{2\pi}\oint_{\gamma_l} \frac{\mathrm{d}q}{3q-4}\frac{q^3+1}{\sqrt{3(q^3+4)(q^3+1)}} = \frac{1}{2\pi}\oint_{\gamma_l'} \frac{x\,\mathrm{d}x}{2\sqrt{27x(x^3-1)}}.$$

Notice that in this case both the reducible and the reduced actions are integrals of the third kind.

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References

[1] Abenda S 1994 Analysis of singularity structures for quasi-integrable Hamiltonian systems *PhD Thesis* Sissa

[2] Abenda S 1997 Asymptotic analysis of time singularities for a class of time dependent Hamiltonians *J. Phys. A: Math. Gen.* **30** 143–71

[3] Abenda S and Bazzani A 1993 Singularity analysis in 2d complexified Hamiltonian systems *Hamiltonian Mechanics: Integrability and Chaotic Behavior (Nato ASI series B)* ed J Seimenis (New York: Plenum)

[4] Adler M and van Moerbeke P 1989 Algebraic completely integrable systems: a systematic approach *Perspectives in Mathematics* (Boston: Academic)

[5] Arnol'd V I, Kozlov V V and Neishtadt A I 1987 Mathematical aspects of classical and celestial mechanics in dynamical systems III *Encyclopaedia of Mathematical Sciences* ed V I Arnol'd (Berlin: Springer)

[6] Appell P and Goursat M 1979 *Theorie des Courbes Algébriques, tome I revue et augmenteé par P Fatou* (New York: Chelsea)

[7] Audin M and Silhol R 1992 *Variétés Abéliennes Réeles et Toupie de Kowalewski* vol 479 (Publication de l'IRMA) p 275
[8] Calogero F and Francoise J P 1997 Hamiltonian character of the motion of the zeros of a polynomial whose coefficients oscillate over time *J. Phys. A: Math. Gen.* **30** 211–18
[9] Dubrovin B A, Krichever I M and Novikov S P 1990 Integrable systems I *Dynamical Systems IV* ed V I Arnol'd and S P Novikov *Encyclopaedia of Mathematical Sciences* vol 4 (Berlin: Springer)
[10] Dubrovin B A 1982 Theta functions and non-linear equations *Russ. Math. Surv.* **36** 11–80
[11] Duistermaat J J 1980 On global action-angle coordinates *Commun. Pure Appl. Math.* **33** 687–706
[12] Francoise J P 1982 *Calculs Explicits d'Action-angle (Nato ASI series)* vol 105 (Université de Montreal)
[13] Griffiths P and Harris J 1978 *Principles of Algebraic Geometry* (New York: Wiley)
[14] Goursat M 1885 Sur la réduction des intégrales hyperelliptiques *Bull. Soc. Math. Fr.* **13** 143–62
[15] Nekhoroscev N N 1972 Action-angle variables and their generalizations *Trans. Moscow Math. Soc.* **26** 180–98
[16] Novikov S and Veselov A 1985 Poisson brackets and complex tori *Proc. Steklov Inst. Math.* **3** 53–65
[17] 1973 *Oeuvres de Paul Painlevé (Éditions du CNRS, Paris)* vol 1 (Paris: Centre National de la Recherche Scientifique) out of print
1974 *Oeuvres de Paul Painlevé (Éditions du CNRS, Paris)* vol 2 (Paris: Centre National de la Recherche Scientifique) out of print
1976 *Oeuvres de Paul Painlevé (Éditions du CNRS, Paris)* vol 3 (Paris: Centre National de la Recherche Scientifique) out of print
[18] Picard E 1882 Sur la réduction du nombre des périodes des intégrales Abéliennes et, en particulier, dans le cas des courbes du second genre *Bull. Soc. Math. Fr.* **11** 19–53
[19] Poincaré H 1883 Sur la réduction des intégrales Abéliennes *Bull. Soc. Math. Fr.* **12** 124–43
[20] Ratiu T and van Moerbecke P 1982 The Lagrange rigid body motion *Ann. Inst. Fourier* **32** 211–34
[21] Siegel C L 1971 *Topics in Complex Function Theory* vol 1 (New York: Wiley–Interscience)
Siegel C L 1971 *Topics in Complex Function Theory* vol 2 (New York: Wiley–Interscience)
Siegel C L 1971 *Topics in Complex Function Theory* vol 3 (New York: Wiley–Interscience)
[22] van Moerbeke P 1987 Introduction to algebraic integrable systems and their Painlevé analysis *Bowdoin AMS Summer Symp. Proc.* (Providence, RI: American Mathematical Society)
[23] Vanhaecke P 1992 Linearizing two-dimensional integrable systems and the construction of action-angle variables *Math. Z.* **211** 265–313
[24] Ziglin S L 1982 Splitting of separatrices, branching of solutions and non-existence of an integral in the dynamics of a solid body *Trans. Moscow Math. Soc.* **41** 283
[25] Ziglin S L 1983 Branching of solutions and the non-existence of first integrals in Hamiltonian mechanics: I *Funct. Anal. Appl.* **16** 181–9